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CALCULATION OF NONSTATIONARY HEAT CONDUCTION IN MULTILAYER OBJECTS WITH BOUNDARY CONDITIONS OF THE THIRD KIND

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Some aspects of the calculation of nonstationary heat conduction in multilayer objects with boundary conditions of the third kind are considered. The homogeneous problem with inhomogeneous boundary conditions is solved for the one-dimensional case. The proposed solution has an explicit form and may be useful in numerical calculations due to the recurrence representation of the basic relations.

Many important practical problems of calculating the temperature field in multilayer objects may be considered as one-dimensional. The literature contains works devoted to solving nonstationary heat conduction problems for a multilayer plate [1, 2] and for a three-layer cylinder [3] and to the general solution of the homogeneous problem in the case of a one-dimensional field [4].

The operator form of a solution cannot, unfortunately, be used in practical calculations because of technical difficulties associated with passing from transforms to an inverse transform. The form of a solution obtained by the Fourier method [4] is also of little use because of the difficulties involved in determining the coefficients when there are three and more layers in an object. Below, the general solution to the nonstationary heat conduction problem for multilayer objects is given which may be employed in practical calculations due to the recurrence form of the basic relations obtained.

The solution of many nonstationary heat conduction problems with arbitrary initial and boundary conditions, is based, as a rule, on a particular solution of the homogeneous problem with inhomogeneous boundary conditions, which is of great interest.

The mathematical formulation of the homogeneous problem of nonstationary heat conduction in a multilayer object in the case of a one-dimensional temperature field is specified by the system of differential equations

$$\frac{\partial T_i(r,\ t)}{\partial t} = a_i \nabla^2 T_i(r,\ t),\ x_{i-1} \leqslant r \leqslant x_i,\ i = 1,\ 2,\ \ldots,\ n, \tag{1}$$

where $T_i(r, t)$, a_i are the temperature field and the thermal diffusivity of the i-th layer; x_0, x_n are the coordinates of upper and lower geometric (free) surfaces of an object.

The boundary conditions on free surfaces $r = x_0$, x_n are determined as

$$\left[T_1(r,\ t) + h_1 \frac{\partial T_1(r,\ t)}{\partial r}\right]_{r=x_0} = \varphi_1,\quad \left[T_n(r,\ t) + h_2 \frac{\partial T_n(r,\ t)}{\partial r}\right]_{r=x_n} = \varphi_2. \tag{2}$$

The boundary conditions of conjugation of temperature fields at an interface of the layers are of the form

$$\left.\begin{aligned} &T_i(r,\ t) = T_{i+1}(r,\ t) \\ &\lambda_i \frac{\partial T_i(r,\ t)}{\partial r} = \lambda_{i+1} \frac{\partial T_{i+1}(r,\ t)}{\partial r} \end{aligned}\right|_{\substack{r=x_i \\ i=1,\ 2,\ldots,n-1}}, \tag{3}$$

where λ_i is the thermal conductivity of the i-th layer.

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The initial conditions in the general form are as follows:

$$T_i(r,\ 0) = f_i(r),\ i = 1,\ 2,\ \ldots,\ n.$$

A solution to the stated problem may be obtained as a sum of particular solutions:

$$T_i(r,\ t) = \varphi_1 + \overset{*}{\psi}_i(r) + w_i(r,\ t),\ i = 1,\ 2,\ \ldots,\ n, \tag{4}$$

where $\overset{*}{\psi}_i(r)$ are functions satisfying the homogeneous differential equation for the stationary case (the Laplace equation), the conjugation conditions (3), and the inhomogeneous boundary conditions

$$\left[\overset{*}{\psi}_1(r) + h_1 \frac{\partial \overset{*}{\psi}_1(r)}{\partial r}\right]_{r=x_0} = 0,\quad \left[\overset{*}{\psi}_n(r) + h_2 \frac{\partial \overset{*}{\psi}_n(r)}{\partial r}\right]_{r=x_n} = \varphi_2 - \varphi_1, \tag{5}$$

and $w_i(r,\ t)$ are functions satisfying the homogeneous differential equation (1) and homogeneous boundary conditions with the following modified initial conditions:

$$w_i(r,\ 0) = f_i(r) - \varphi_1 - \overset{*}{\psi}_i(r),\ i = 1,\ 2,\ \ldots,\ n. \tag{6}$$

In the general case, the functions $\overset{*}{\psi}_i(r)$ and $w_i(r,\ t)$ are determined by the following expressions:

$$\overset{*}{\psi}_i(r) = [\varphi_2 - \varphi_1]\, \overset{*}{\alpha}_i\, [\xi(r) + \overset{*}{\beta}_i],\ i = 1,\ 2,\ \ldots,\ n, \tag{7}$$

for the given class of problems, where $\overset{*}{\alpha}_i$, $\overset{*}{\beta}_i$ are calculated using the recurrence formulas

$$\overset{*}{\beta}_1 = -[\xi(x_0) + h_1 \xi'(x_0)],$$

$$\overset{*}{\beta}_i = [\lambda_i/\lambda_{i-1}]\,[\xi(x_i) + \overset{*}{\beta}_{i-1}] - \xi(x_i),\ i = 2,\ 3,\ \ldots,\ n, \tag{8}$$

$$\overset{*}{\alpha}_i = [\lambda_n/\lambda_i]/[\xi(x_n) + \overset{*}{\beta}_n + h_2 \xi'(x_n)],\ i = 1,\ 2,\ \ldots,\ n,$$

$$w_i(r,\ t) = \sum_{m=0}^{\infty} A_{1,m} \overset{*}{F}_{i,m}(\mu_{i,m} r) \exp[-\mu_{i,m}^2 a_i t],\ i = 1,\ 2,\ \ldots,\ n, \tag{9}$$

where

$$\overset{*}{F}_{i,m}(\mu_{i,m} r) = \left[\prod_{k=1}^{i} Z_k\right][Y_1(\mu_{i,m} r) + B_{i,m} Y_2(\mu_{i,m} r)], \tag{10}$$

$$B_{1,m} = -\frac{Y_1(\mu_{1,m} x_0) + h_1 Y_1'(\mu_{1,m} x_0)}{Y_2(\mu_{1,m} x_0) + h_1 Y_2'(\mu_{1,m} x_0)}, \tag{11}$$

$$B_{i,m} = \frac{\lambda_i \dfrac{Y_1'(\mu_{i,m} x_{i-1})}{Y_1(\mu_{i,m} x_{i-1})} - \lambda_{i-1} \dfrac{Y_1'(\mu_{i-1,m} x_{i-1}) + B_{i-1,m} Y_2'(\mu_{i-1,m} x_{i-1})}{Y_1(\mu_{i-1,m} x_{i-1}) + B_{i-1,m} Y_2(\mu_{i-1,m} x_{i-1})}}{\lambda_i \dfrac{Y_2'(\mu_{i,m} x_{i-1})}{Y_2(\mu_{i,m} x_{i-1})} - \lambda_{i-1} \dfrac{Y_1'(\mu_{i-1,m} x_{i-1}) + B_{i-1,m} Y_2'(\mu_{i-1,m} x_{i-1})}{Y_1(\mu_{i-1,m} x_{i-1}) + B_{i-1,m} Y_2(\mu_{i-1,m} x_{i-1})}} \times$$

$$\times \frac{Y_1(\mu_{i,m} x_{i-1})}{Y_2(\mu_{i,m} x_{i-1})},\ i = 2,\ 3,\ \ldots,\ n, \tag{12}$$

$$Z_1 = 1,\ Z_i = \frac{Y_1(\mu_{i-1,m} x_{i-1}) + B_{i-1,m} Y_2(\mu_{i-1,m} x_{i-1})}{Y_1(\mu_{i,m} x_{i-1}) + B_{i,m} Y_2(\mu_{i,m} x_{i-1})},\ i = 2,\ 3,\ \ldots,\ n, \tag{13}$$

$$A_{1,m} = \sum_{i=1}^{n} \frac{\lambda_i}{a_i} \int_{x_{i-1}}^{x_i} w_i(r,\ 0)\, G(r)\, \overset{*}{F}_{i,m}(\mu_{i,m} r)\, dr \Big/ \sum_{i=1}^{n} \overset{*}{J}{}_i^2, \tag{14}$$

$$\overset{*}{J}{}_i^2 = \frac{\lambda_i}{a_i} \int_{x_{i-1}}^{x_i} G(r)\, \overset{*}{F}{}_{i,m}^2(\mu_{i,m} r)\, dr, \tag{15}$$

$\mu_{i,m} = \mu_{n,m}\sqrt{a_n/a_i}$, where $\mu_{n,m}$ are the eigenvalues of the problem, determined according to the equation

$$Y_1(\mu_{n,m}x_n) + h_2Y_1'(\mu_{n,m}x_n) + B_{n,m}[Y_2(\mu_{n,m}x_n) + h_2Y_2'(\mu_{n,m}x_n)],\ m=0,\ 1,\ \ldots \tag{16}$$

The weighting function G(r) and the specific form of the functions $\xi(r)$, $\overset{*}{F}_i(\mu_i r)$ are uniquely determined, depending on the chosen coordinate system, by the following expressions:

in a rectangular coordinate system:

$$G(r) = 1,\ \xi(r) = r,\ Y_1(\mu r) = \sin(\mu r),\ Y_2(\mu r) = \cos(\mu r);$$

in a cylindrical coordinate system:

$$G(r) = r,\ \xi(r) = \ln r,\ Y_1(\mu r) = J_0(\mu r),\ Y_2(\mu r) = N_0(\mu r);$$

in a spherical coordinate system:

$$G(r) = r^2,\ \xi(r) = 1/r,\ Y_1(\mu r) = \frac{1}{r}\sin(\mu r),\ Y_2(\mu r) = \frac{1}{r}\cos(\mu r).$$

NOTATION

T(r, t), one-dimensional temperature field; t, time coordinate; r, spatial coordinate; a, thermal diffusivity; λ, thermal conductivity.

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